

# ON THE BETA-NUMBER OF FORESTS WITH ISOMORPHIC COMPONENTS

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*Dedicated to the memory of Professor Mirka Miller*

ABSTRACT. The beta-number,  $\beta(G)$ , of a graph  $G$  is defined to be either the smallest positive integer  $n$  for which there exists an injective function  $f : V(G) \rightarrow \{0, 1, \dots, n\}$  such that each  $uv \in E(G)$  is labeled  $|f(u) - f(v)|$  and the resulting set of edge labels is  $\{c, c+1, \dots, c+|E(G)|-1\}$  for some positive integer  $c$  or  $+\infty$  if there exists no such integer  $n$ . If  $c = 1$ , then the resulting beta-number is called the strong beta-number of  $G$  and is denoted by  $\beta_s(G)$ . In this paper, we show that if  $G$  is a bipartite graph and  $m$  is odd, then  $\beta(mG) \leq m\beta(G) + m - 1$ . This leads us to conclude that  $\beta(mG) = m|V(G)| - 1$  if  $G$  has the additional property that  $G$  is a graceful nontrivial tree. In addition to these, we examine the (strong) beta-number of forests whose components are isomorphic to either paths or stars.

## 1. INTRODUCTION

Terms and notation not defined below follow that used in the book by Chartrand and Lesniak [3]. All graphs considered in this paper are finite and undirected without loops or multiple edges. The vertex set and edge set of a graph  $G$  is denoted by  $V(G)$  and  $E(G)$ , respectively. Let  $G_1$  and  $G_2$  be vertex-disjoint graphs. Then the *union* of  $G_1$  and  $G_2$ , denoted by  $G_1 \cup G_2$ , is the graph with  $V(G_1 \cup G_2) = V(G_1) \cup V(G_2)$  and  $E(G_1 \cup G_2) = E(G_1) \cup E(G_2)$ . If  $G_1, G_2, \dots, G_m$  are pairwise vertex-disjoint graphs that are isomorphic to  $G$ , then we write  $mG$  for  $G_1 \cup G_2 \cup \dots \cup G_m$ . As usual, we denote a path with  $n$  vertices and a star with  $n+1$  vertices by  $P_n$  and  $S_n$ , respectively.

For integers  $a$  and  $b$  with  $a \leq b$ , we will write  $[a, b]$  for the set  $\{x \in \mathbb{Z} : a \leq x \leq b\}$ , where  $\mathbb{Z}$  denotes the set of integers. On the other hand, if  $a > b$ , then we treat  $[a, b]$  as an empty set. If such situations occur in particular formulas for a given vertex labeling, then we ignore the corresponding portions of formulas.

The type of graph labelings that have received the most attention over the years was introduced by Rosa [13] in 1967 who called them  $\beta$ -valuations. For a graph  $G$  of size  $q$ , an injective function  $f : V(G) \rightarrow [0, q]$  is called a  $\beta$ -valuation if each  $uv \in E(G)$  is labeled  $|f(u) - f(v)|$  and the resulting set of edge labels is  $[1, q]$ . Such a valuation is now commonly known as a *graceful labeling* (the term was coined by Golomb [8] in 1972) and a graph with a graceful labeling is called *graceful*. The concept of  $\alpha$ -valuations (a particular type of graceful labeling) was also introduced

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by Rosa [13] as a tool for decomposing the complete graph into isomorphic sub-graphs. A graceful labeling  $f$  is called an  $\alpha$ -valuation if there exists an integer  $\lambda$  so that  $\min\{f(u), f(v)\} \leq \lambda < \max\{f(u), f(v)\}$  for each  $uv \in E(G)$ .

The *gracefulness*,  $\text{grac}(G)$ , of a graph  $G$  is the smallest positive integer  $n$  for which there exists an injective function  $f : V(G) \rightarrow [0, n]$  such that each  $uv \in E(G)$  is labeled  $|f(u) - f(v)|$  and the resulting set of edge labels consists of distinct integers. If  $G$  is a graph of size  $q$  with  $\text{grac}(G) = q$ , then  $G$  is graceful. Thus, the gracefulness of a graph  $G$  is a measure of how close  $G$  is to being graceful. This definition first appeared in a paper by Golomb [8].

Motivated by the concept of the gracefulness of a graph, Ichishima et al. [9] introduced the (strong) beta-number of a graph. The *beta-number*,  $\beta(G)$ , of a graph  $G$  with  $q$  edges is either the smallest positive integer  $n$  for which there exists an injective function  $f : V(G) \rightarrow [0, n]$  such that each  $uv \in E(G)$  is labeled  $|f(u) - f(v)|$  and the resulting set of edge labels is  $[c, c + q - 1]$  for some positive integer  $c$  or  $+\infty$  if there exists no such integer  $n$ . If  $c = 1$ , then the resulting beta-number is called the *strong beta-number* of  $G$  and is denoted by  $\beta_s(G)$ . These parameters can be regarded as measures of how close a graph is to being graceful.

To present the new results contained in this paper, the following lemmas taken from [9] will prove to be useful.

**Lemma 1.** *For every graph  $G$  of order  $p$  and size  $q$ ,*

$$\max\{p - 1, q\} \leq \text{grac}(G) \leq \beta(G) \leq \beta_s(G).$$

**Lemma 2.** *For every two positive integers  $m$  and  $n$ ,*

$$\beta(S_m \cup S_n) = \beta_s(S_m \cup S_n) = \begin{cases} m + n + 1 & \text{if } mn \text{ is even,} \\ m + n + 2 & \text{if } mn \text{ is odd.} \end{cases}.$$

In Section 2, we prove that if  $G$  is a bipartite graph and  $m$  is odd, then  $\beta(mG) \leq m\beta(G) + m - 1$ . This leads us to conclude that if  $T$  is a graceful nontrivial tree of order  $p$  and  $m$  is odd, then  $\beta(mT) = mp - 1$ . In Section 3, we compute the (strong) beta-number of forests whose components are all paths of order 2. In Section 4, we examine the (strong) beta-number of forests that all components are isomorphic to either paths or stars. In Section 5, we propose new conjectures on the (strong) beta-number of forests.

## 2. MAIN RESULTS

According to the survey on graph labelings by Gallian [7], various classes of bipartite graphs have been proved to admit graceful labelings. Furthermore, a number of techniques to construct trees from smaller ones with graceful labelings have been shown to yield graceful labelings in the resulting trees. In this section, we thus compute the beta-number for an odd number of copies of an isomorphic tree with a graceful labeling. To do this, we start with the following result.

**Theorem 1.** *If  $G$  is a bipartite graph and  $m$  is odd, then*

$$\beta(mG) \leq m\beta(G) + m - 1.$$

*Proof.* Let  $G$  be a bipartite graph of size  $q$ . First, notice that the result is trivial when  $\beta(G) = +\infty$ . Thus, assume that  $\beta(G) = n$  for some positive integer  $n$ . Then there exists an injective function  $f : V(G) \rightarrow [0, n]$  such that

$$\{|f(u) - f(v)| : uv \in E(G)\} = [c, c + q - 1]$$

for some positive integer  $c$ . If  $G$  has the partite sets  $U$  and  $V$ , then let  $E(G) = UV$ , where the juxtaposition of two partite sets denotes the edges between those two sets. Now, define  $H \cong mG$  to be the graph with

$$V(H) = \bigcup_{i=1}^m (U_i \cup V_i) \text{ and } E(H) = \bigcup_{i=1}^m U_i V_i,$$

where  $x_i \in X_i$  ( $i \in [1, m]$ ) if and only if  $x \in X$  ( $X$  is one of the sets  $U$  or  $V$ ).

Next, consider the vertex labeling  $g : V(H) \rightarrow [0, mn + m - 1]$  such that

$$g(x_i) = \begin{cases} mf(x) + i - 1 & \text{if } x \in U \text{ and } i \in [1, m], \\ mf(x) + (i - 1)/2 & \text{if } x \in V \text{ and } i \text{ is odd,} \\ mf(x) + (m - 1 + i)/2 & \text{if } x \in V \text{ and } i \text{ is even.} \end{cases}$$

This leads to conclude that  $\beta(mG) \leq mn + m - 1$  when  $m$  is odd. To verify this, notice that for each  $uv \in E(G)$ , where  $u \in U$  and  $v \in V$  such that  $f(u) > f(v)$ , we have

$$|g(u) - g(v)| = \begin{cases} m(f(u) - f(v)) + (i - 1)/2 & \text{if } i \text{ is odd,} \\ m(f(u) - f(v)) + (i - 1 - m)/2 & \text{if } i \text{ is even.} \end{cases}$$

Notice also that for each  $uv \in E(G)$ , where  $u \in U$  and  $v \in V$  such that  $f(u) < f(v)$ , we have

$$|g(u) - g(v)| = \begin{cases} m(f(v) - f(u)) - (i - 1)/2 & \text{if } i \text{ is odd,} \\ m(f(v) - f(u)) - (i - 1 - m)/2 & \text{if } i \text{ is even.} \end{cases}$$

Consequently, we have

$$\{|g(u) - g(v)| : uv \in E(H)\} = [c', c' + mq - 1],$$

where  $c' = (2mc - m + 1)/2$ . Finally, to see that

$$\{g(x) : x \in V(H)\} \subseteq [0, mn + m - 1],$$

notice that for each  $x \in V(G)$ , we have

$$\{f(x) : x \in V(G)\} \subseteq [0, n] \text{ and } \bigcup_{i=1}^m \{g(x_i)\} \subseteq \bigcup_{i=1}^m \{mf(x) + i - 1\}.$$

□

Figuerola-Centeno et al. [6] introduced the  $\otimes_h$ -product of digraphs as follows. Let  $D$  be a digraph and let  $\Gamma = \{F_1, F_2, \dots, F_n\}$  be a family of digraphs such that  $V(F_i) = V$  for each  $i \in [1, n]$ . Consider a function  $h : E(D) \rightarrow \Gamma$ . Then the *product*  $D \otimes_h \Gamma$  is defined to be the digraph with vertex set  $V(D) \times V$  and  $((a, b), (c, d)) \in E(D \otimes_h \Gamma)$  if and only if  $(a, c) \in E(D)$  and  $(b, d) \in E(h(a, c))$ . Using this concept together with the techniques described in [12], it is possible to provide an alternative proof of Theorem 1. However, this approach requires to introduce a considerable amount of machinery. Since the proof presented in this paper is clear and easy to understand, we omit the alternative approach mentioned above.

If  $G$  is a graceful bipartite graph of size  $q$ , then  $\beta(G) = q$ . This together with Theorem 1 gives us the following result.

**Corollary 1.** *If  $G$  is a graceful bipartite graph of size  $q$ , then*

$$\beta(mG) \leq mq + m - 1,$$

*where  $m$  is odd.*

For a graceful nontrivial tree, we have the following result, which shows that in this case, the bound provided in Corollary 1 is sharp.

**Corollary 2.** *If  $T$  is a graceful nontrivial tree of order  $p$ , then*

$$\beta(mT) = mp - 1,$$

*where  $m$  is odd.*

*Proof.* Let  $T$  be a graceful nontrivial tree of order  $p$ , and assume that  $m$  is odd. Since  $T$  has size  $p - 1$ , it follows from Corollary 1 that  $\beta(mT) \leq mp - 1$ . Further, the reverse inequality is easily obtained from Lemma 1.  $\square$

The preceding result is the best possible in the sense that  $m$  cannot be even, since every star  $S_n$  is graceful (see [13]) and, by Lemma 2, we have  $\beta(S_m \cup S_n) = m + n + 2$  when  $mn$  is odd. Thus, for every positive integer  $n$ , we have  $\beta(2S_{2n-1}) = 4n$ , but  $2|V(S_{2n-1})| - 1 = 4n - 1$ .

We end this section with the following result, which is a direct consequence of Theorem 1 and Lemma 1.

**Corollary 3.** *If  $F$  is a forest of order  $p$  such that  $\beta(F) = p - 1$ , then*

$$\beta(mF) = mp - 1,$$

*where  $m$  is odd.*

### 3. CONNECTIONS WITH SKOLEM SEQUENCES

To present our results in this section, we utilize the notion of a Skolem sequence and its generalization, which has been widely used in the construction of combinatorial designs. The motivation for these sequences came from the area of balanced incomplete block designs, in particular, Steiner Triple Systems (see [15]). These systems are used for, (among other things), interference resistant message code for missile guidance systems (see [4]).

A *Skolem sequence* of order  $m$  is a sequence  $S = (s_1, s_2, \dots, s_{2m})$  of  $2m$  integers satisfying the conditions:

- (S1) for every  $k \in [1, m]$ , there exist exactly two subscripts  $i(k)$  and  $j(k)$  such that  $s_{i(k)} = s_{j(k)} = k$ , and
- (S2) if  $s_{i(k)} = s_{j(k)} = k$  with  $i(k) < j(k)$ , then  $j(k) - i(k) = k$ .

The existence of Skolem sequences was settled in 1958 by Skolem [16] as we state in the following theorem.

**Theorem 2.** *A Skolem sequence of order  $m$  exists if and only if  $m \equiv 0$  or  $1 \pmod{4}$ .*

For  $m \equiv 2$  or  $3 \pmod{4}$ , the natural alternative is a hooked Skolem sequence. A *hooked Skolem sequence* of order  $m$  is a sequence  $S = (s_1, s_2, \dots, s_{2m+1})$  of  $2m + 1$  integers satisfying conditions (S1) and (S2) of the definition of a Skolem sequence and

- (S3)  $s_{2m} = 0$ .

The existence of hooked Skolem sequences was settled in 1961 by O'Keefe [10] as we state in the following theorem.

**Theorem 3.** *A hooked Skolem sequence of order  $m$  exists if and only if  $m \equiv 2$  or  $3 \pmod{4}$ .*

It is worth to mention that López and Muntaner-Batle [11] recently have found strong relations between super edge-magic labelings of graphs introduced independently in [1] and [5], and Skolem type of sequences. Their results can be used to improve the previously known bounds for the number of Skolem type of sequences. Hence, graph labelings and Skolem type of sequences are tightly related.

With the aid of Theorems 2 and 3, we are now able to provide the following result.

**Theorem 4.** *For every positive integer  $m$ ,*

$$\beta_s(mP_2) = \begin{cases} 2m - 1 & \text{if } m \equiv 0 \text{ or } 1 \pmod{4}, \\ 2m & \text{if } m \equiv 2 \text{ or } 3 \pmod{4}. \end{cases}$$

*Proof.* Let  $F \cong mP_2$  be the forest with  $V(F) = \{x_k : k \in [1, m]\} \cup \{y_k : k \in [1, m]\}$  and  $E(F) = \{x_k y_k : k \in [1, m]\}$ . Since  $|V(F)| = 2m$  and  $|E(F)| = m$ , it follows from Lemma 1 that  $\beta_s(F) \geq 2m - 1$  for every positive integer  $m$ .

First, assume that  $m \equiv 0$  or  $1 \pmod{4}$ . By Theorem 2, there exists a Skolem sequence of order  $m$ . That is, for each  $k \in [1, m]$ , there exist exactly two subscripts  $i(k)$  and  $j(k)$  satisfying all the conditions of the definition of a Skolem sequence. Thus, the vertex labeling  $f : V(F) \rightarrow [0, 2m - 1]$  such that

$$f(x_k) = i(k) - 1 \text{ and } f(y_k) = j(k) - 1 \quad (k \in [1, m])$$

is a bijective function and satisfies that

$$\begin{aligned} & \{|f(x_k) - f(y_k)| : k \in [1, m]\} \\ &= \{|i(k) - j(k)| : s_{i(k)} = s_{j(k)} = k \text{ and } k \in [1, m]\} = [1, |E(F)|], \end{aligned}$$

since  $i(k), j(k) \in [1, 2m]$  and  $i(k) < j(k)$  for all  $k \in [1, m]$ , and  $|E(F)| = m$ . Therefore,  $\beta_s(F) \leq 2m - 1$ , which leads to conclude that  $\beta_s(F) = 2m - 1$  when  $m \equiv 0$  or  $1 \pmod{4}$ .

For  $m \equiv 2$  or  $3 \pmod{4}$ , suppose, to the contrary, that  $\beta_s(F) = 2m - 1$ . Since  $|V(F)| = 2m$ , it follows that there exists a bijective function  $g : V(F) \rightarrow [0, 2m - 1]$  such that  $\{|g(x_k) - g(y_k)| : k \in [1, m]\} = [1, m]$ . This produces a Skolem sequence of order  $m$  by letting

$$i(k) = g(x_k) + 1 \text{ and } j(k) = g(y_k) + 1 \quad (k \in [1, m]),$$

which is impossible by Theorem 2. Thus,  $\beta_s(F) \geq 2m$  when  $m \equiv 2$  or  $3 \pmod{4}$ . On the other hand, it follows from Theorem 3 that there exists a hooked Skolem sequence of order  $m$ . That is, for every  $k \in [1, m]$ , there exists exactly two subscripts  $i(k)$  and  $j(k)$  satisfying all the conditions of the definition of a hooked Skolem sequence. Thus, the vertex labeling  $h : V(F) \rightarrow [0, 2m]$  such that

$$h(x_k) = i(k) - 1 \text{ and } h(y_k) = j(k) - 1 \quad (k \in [1, m])$$

is an injective function and satisfies that

$$\begin{aligned} & \{|h(x_k) - h(y_k)| : k \in [1, m]\} \\ &= \{|i(k) - j(k)| : s_{i(k)} = s_{j(k)} = k \text{ and } k \in [1, m]\} = [1, |E(F)|], \end{aligned}$$

since  $i(k), j(k) \in [1, 2m+1]$  and  $i(k) < j(k)$  for all  $k \in [1, m]$ , and  $|E(F)| = m$ . Therefore,  $\beta_s(F) \leq 2m$ , which leads to conclude that  $\beta_s(F) = 2m$  when  $m \equiv 2$  or  $3 \pmod{4}$ .  $\square$

With the aid of Corollary 2 and Theorem 4, we are now able to present the following result.

**Corollary 4.** *For every positive integer  $m$ ,*

$$\beta(mP_2) = \begin{cases} 2m-1 & \text{if } m \not\equiv 2 \pmod{4}, \\ 2m & \text{if } m \equiv 2 \pmod{4}. \end{cases}$$

*Proof.* Let  $F \cong mP_2$  be the forest as in the proof of Theorem 4. Since  $P_2$  is clearly graceful, it follows from Corollary 2 that  $\beta(F) = 2m-1$  for  $m \equiv 1$  or  $3 \pmod{4}$ . It also follows from Lemma 1 and Theorem 4 that  $\beta(F) = 2m-1$  for  $m \equiv 0 \pmod{4}$  and  $\beta(F) \leq 2m$  for  $m \equiv 2 \pmod{4}$ .

Next, assume that  $m \equiv 2 \pmod{4}$ , and suppose, to the contrary, that  $\beta(F) = 2m-1$ . Since  $|V(F)| = 2m$ , it certainly follows that there exists a bijective function  $f : V(F) \rightarrow [0, 2m-1]$  such that  $\{|f(x_i) - f(y_i)| : i \in [1, m]\} = [c, c+m-1]$  for some positive integer  $c$ . Thus, the sum of induced edge labels of  $F$  is

$$\begin{aligned} \sum_{i=1}^m |f(x_i) - f(y_i)| &\equiv \sum_{i=1}^m (f(x_i) - f(y_i)) \pmod{2} \\ &\equiv \sum_{i=1}^m (f(x_i) + f(y_i)) \pmod{2} \\ &\equiv m(2m-1) \equiv 0 \pmod{2}. \end{aligned}$$

That is, the sum of induced edge labels of  $F$  is even. However, the sum of the edge labels is

$$\sum_{i=0}^{m-1} (i+c) = m(m+2c-1)/2,$$

which is odd for any positive integer  $c$ . This produces a contradiction and thus  $\beta(F) \geq 2m$  for  $m \equiv 2 \pmod{4}$ , which leads to conclude that  $\beta(F) = 2m$  for  $m \equiv 2 \pmod{4}$ .  $\square$

#### 4. FURTHER RESULTS ON FORESTS WITH ISOMORPHIC COMPONENTS

As we have already mentioned in Section 2, we cannot extend Corollary 2 to the case that the number of components is even. However, in this section, we are able to determine the (strong) beta-number for some classes of forests with even number of isomorphic components.

Before presenting our next result, we briefly discuss some previously known results on  $\alpha$ -valuations and graceful labelings of paths. We first mention the result obtained by Rosa [14].

**Theorem 5.** *For any positive integer  $n$  and any vertex  $v$  of the path  $P_n$ , there exists a graceful labeling  $f$  of  $P_n$  with  $f(v) = 0$ . Moreover, an  $\alpha$ -valuation of  $P_n$  with  $f(v) = 0$  exists if and only if  $v$  is not the central vertex of  $P_5$ .*

When concerning a graceful labeling of a path, Cattell [2] has shown that one has almost complete freedom to choose a particular label  $i$  for any given vertex  $v$ . In particular, he proved the following result.

**Theorem 6.** *Let  $v$  be any vertex of the nontrivial path  $P_n$ . Then there exists a graceful labeling  $f$  of  $P_n$  with  $f(v) = i$  for any  $i \in [1, n-1]$  unless  $n \equiv 3 \pmod{4}$  or  $n \equiv 1 \pmod{12}$ ;  $v$  is in the smaller of the two partite sets of vertices, and  $i = (n-1)/2$ .*

As an immediate consequence of Theorem 6, we obtain the following result.

**Corollary 5.** *Let  $v$  be an end-vertex of the path  $P_n$  ( $n \geq 3$ ). Then there exists a graceful labeling  $f$  of  $P_n$  with  $f(v) \leq \lfloor (n-1)/2 \rfloor - 1$ .*

With the aid of Theorem 5 and Corollary 5, we are able to determine the (strong) beta-number of the forest whose components are two copies of nontrivial paths as our next result indicates. Our proof uses the concept of induced subgraph. Let  $S$  be a nonempty set of vertices of a graph  $G$ . The *subgraph induced by  $S$*  is the maximal subgraph of  $G$  with vertex set  $S$ , and is denoted by  $\langle S \rangle$ , that is, contains precisely those edges of  $G$  joining two vertices in  $S$ . A subgraph  $H$  of a graph  $G$  is an *induced subgraph* if  $H \cong \langle S \rangle$  for some nonempty set  $S$  of vertices of  $G$ .

**Theorem 7.** *For every integer  $n \geq 2$ ,*

$$\beta(2P_n) = \beta_s(2P_n) = \begin{cases} 2n & \text{if } n = 2, \\ 2n - 1 & \text{if } n \geq 3. \end{cases}$$

*Proof.* For  $n = 2$ , the result easily follows from Theorem 4 and Corollary 4. It also follows from Lemma 2 that  $\beta(S_m \cup S_n) = \beta_s(S_m \cup S_n) = m + n + 1$  when  $mn$  is even. Combining this with the fact that  $2S_2 \cong 2P_3$ , we have the result for  $n = 3$ . Let  $F \cong 2P_n$  be the forest with

$$V(F) = \{x_i : i \in [1, n]\} \cup \{y_i : i \in [1, n]\}$$

and

$$E(F) = \{x_i x_{i+1} : i \in [1, n-1]\} \cup \{y_i y_{i+1} : i \in [1, n-1]\}.$$

Then, in light of Lemma 1 and Table 2, it suffices to show that  $\beta_s(F) \leq 2n-1$  when  $n \geq 12$ . Thereby, we first consider the vertex labels of the induced subgraphs of  $F$ . Let  $F_1$  be the induced subgraph of  $F$  with  $V(F_1) = \{x_i : i \in [1, n]\}$ . Then  $F_1 \cong P_n$ , and Theorem 5 guarantees the existence of an  $\alpha$ -valuation  $f_1$  of  $F_1$  with  $f_1(x_1) = 0$ . If we define the induced subgraph  $F_2$  of  $F$  with  $V(F_2) = \{y_i : i \in [1, \lceil n/2 \rceil + 1]\}$ , then  $F_2 \cong P_{\lceil n/2 \rceil + 1}$ , and Corollary 5 guarantees the existence of a graceful labeling  $f_2$  of  $F_2$  with  $f_2(y_{\lceil n/2 \rceil + 1}) = \lfloor n/4 \rfloor - 1$ . Let  $F_3$  be the induced subgraph of  $F$  with  $V(F_3) = \{x_i : i \in [\lceil n/2 \rceil + 2, n]\}$ . Then  $F_3 \cong P_{\lfloor n/2 \rfloor - 1}$ , and Theorem 5 guarantees the existence of an  $\alpha$ -valuation  $f_3$  of  $F_3$  with  $f_3(y_{\lceil n/2 \rceil + 2}) = 0$ .

We now consider the vertex labeling  $g : V(F) \rightarrow [0, 2n-1]$  such that

$$g(x_j) = \begin{cases} 1 + f_1(x_j) & \text{if } j = 2i - 1 \text{ and } i \in [1, \lceil n/2 \rceil], \\ n + f_1(x_j) & \text{if } j = 2i \text{ and } i \in [1, \lceil n/2 \rceil], \end{cases}$$

and

$$g(y_j) = \begin{cases} n - \lfloor n/4 \rfloor + f_2(y_j) & \text{if } j \in [1, \lceil n/2 \rceil + 1], \\ 0 & \text{if } j = \lceil n/2 \rceil + 2, \\ n - 1 - f_3(y_j) & \text{if } j = \lceil n/2 \rceil + 1 + 2i \text{ and } i \in [1, \lceil (n-1)/4 \rceil - 1], \\ \lceil 3n/2 \rceil - f_3(y_j) & \text{if } j = \lceil n/2 \rceil + 2 + 2i \text{ and } i \in [1, \lfloor n/4 \rfloor - 1]. \end{cases}$$

Then  $g$  is a bijective function. To show this, we first compute the vertex labels of  $F_1$ . Since  $f_1$  is an  $\alpha$ -valuation of  $F_1$  with  $f_1(x_1) = 0$ , it follows that

$$\{f_1(x_{2i-1}) : i \in [1, \lceil n/2 \rceil]\} = [0, \lceil n/2 \rceil - 1]$$

and

$$\{f_1(x_{2i}) : i \in [1, \lfloor n/2 \rfloor]\} = [\lceil n/2 \rceil, n-1].$$

This in turn implies that

$$\{g(x_{2i-1}) : i \in [1, \lceil n/2 \rceil]\} = [1, \lceil n/2 \rceil]$$

and

$$\{g(x_{2i}) : i \in [1, \lfloor n/2 \rfloor]\} = [n + \lceil n/2 \rceil, 2n-1] = [\lceil 3n/2 \rceil, 2n-1],$$

which implies that

$$\{g(v) : v \in V(F_1)\} = [1, \lceil n/2 \rceil] \cup [\lceil 3n/2 \rceil, 2n-1].$$

We next compute the vertex labels of  $F_2$ . Since  $f_2$  is a graceful labeling of  $F_2$ , it follows that

$$\{f_2(y_i) : i \in [1, \lceil n/2 \rceil + 1]\} = [0, \lceil n/2 \rceil],$$

which implies that

$$\{g(y_i) : i \in [1, \lceil n/2 \rceil + 1]\} = [n - \lfloor n/4 \rfloor, n - \lfloor n/4 \rfloor + \lceil n/2 \rceil],$$

that is,

$$\{g(v) : v \in V(F_2)\} = [n - \lfloor n/4 \rfloor, n - \lfloor n/4 \rfloor + \lceil n/2 \rceil].$$

It remains to compute the vertex labels of  $F_3$ . Since  $f_3$  is an  $\alpha$ -valuation of  $F_3$  with  $f_3(y_{\lceil n/2 \rceil + 2}) = 0$ , it follows that

$$\{f_3(y_{\lceil n/2 \rceil + 1 + 2i}) : i \in [1, \lceil (n-1)/4 \rceil - 1]\} = [\lfloor n/4 \rfloor, n - \lceil n/2 \rceil - 2]$$

and

$$\{f_3(y_{\lceil n/2 \rceil + 2 + 2i}) : i \in [1, \lceil (n-1)/4 \rceil - 1]\} = [1, \lfloor n/4 \rfloor - 1].$$

This in turn implies that

$$\begin{aligned} \{g(y_{\lceil n/2 \rceil + 1 + 2i}) : i \in [1, \lceil n/4 \rceil - 1]\} &= [n - \lfloor n/2 \rfloor + 1, n - \lfloor n/4 \rfloor - 1] \\ &= [\lceil n/2 \rceil + 1, n - \lfloor n/4 \rfloor - 1] \end{aligned}$$

and

$$\begin{aligned} \{g(y_{\lceil n/2 \rceil + 2 + 2i}) : i \in [1, \lfloor n/4 \rfloor - 1]\} &= [\lceil 3n/2 \rceil - \lfloor n/4 \rfloor + 1, \lceil 3n/2 \rceil - 1] \\ &= [n - \lfloor n/4 \rfloor + \lceil n/2 \rceil + 1, \lceil 3n/2 \rceil - 1]. \end{aligned}$$

This together with  $g(y_{\lceil n/2 \rceil + 2}) = 0$  implies that

$$\begin{aligned} \{g(v) : v \in V(F_3)\} &= \{0\} \cup [\lceil n/2 \rceil + 1, n - \lfloor n/4 \rfloor - 1] \\ &\quad \cup [n - \lfloor n/4 \rfloor + \lceil n/2 \rceil + 1, \lceil 3n/2 \rceil - 1]. \end{aligned}$$

Thus, we have

$$\{g(v) : v \in V(F)\} = [0, 2n-1],$$

which shows that  $g$  is a bijective function.

We finally compute the induced edge labels by ascending order. Since  $f_2$  is a graceful labeling of  $F_2$ , it follows that

$$\{|f_2(u) - f_2(v)| : uv \in E(F_2)\} = [1, \lceil n/2 \rceil],$$

which implies that

$$\{|g(u) - g(v)| : uv \in E(F_2)\} = [1, \lceil n/2 \rceil].$$

Next, notice that  $f_3(y_{\lceil n/2 \rceil + 3}) = n - \lceil n/2 \rceil - 2$ . This implies that

$$g(y_{\lceil n/2 \rceil + 3}) = n - 1 - f_3(y_{\lceil n/2 \rceil + 3}) = \lceil n/2 \rceil + 1.$$



This together with  $g(y_{\lceil n/2 \rceil + 2}) = 0$  implies that

$$|g(y_{\lceil n/2 \rceil + 3}) - g(y_{\lceil n/2 \rceil + 2})| = \lceil n/2 \rceil + 1.$$

For all  $uv \in E(F_3) - \{y_{\lceil n/2 \rceil + 2}y_{\lceil n/2 \rceil + 3}\}$ , where  $u \in \{y_{\lceil n/2 \rceil + 2 + 2i} : i \in [1, \lfloor n/4 \rfloor - 1]\}$  and  $v \in \{y_{\lceil n/2 \rceil + 1 + 2i} : i \in [1, \lceil (n-1)/4 \rceil - 1]\}$ , we know from the vertex labels computed in the above that  $f_3(u) < f_3(v)$ . This implies that  $g(u) > g(v)$  and

$$\begin{aligned} |g(u) - g(v)| &= g(u) - g(v) \\ &= \lceil 3n/2 \rceil - n + 1 - (f_3(u) - f_3(v)) \end{aligned}$$

for all  $uv \in E(F_3) - \{y_{\lceil n/2 \rceil + 2}y_{\lceil n/2 \rceil + 3}\}$ , where  $u \in \{y_{\lceil n/2 \rceil + 2 + 2i} : i \in [1, \lfloor n/4 \rfloor - 1]\}$  and  $v \in \{y_{\lceil n/2 \rceil + 1 + 2i} : i \in [1, \lceil (n-1)/4 \rceil - 1]\}$ . It is now easy to see that

$$\{f_3(u) - f_3(v) : uv \in E(F_3) - \{y_{\lceil n/2 \rceil + 2}y_{\lceil n/2 \rceil + 3}\}\} = [1, \lfloor n/2 \rfloor - 3],$$

since  $f_3$  is an  $\alpha$ -valuation of  $F_3$  with  $f_3(y_{\lceil n/2 \rceil + 2}) = 0$ . From this and the last equation on the induced edge labels  $|g(u) - g(v)|$ , where  $uv \in E(F_3) - \{y_{\lceil n/2 \rceil + 2}y_{\lceil n/2 \rceil + 3}\}$ , we obtain

$$\{|g(u) - g(v)| : uv \in E(F_3) - \{y_{\lceil n/2 \rceil + 2}y_{\lceil n/2 \rceil + 3}\}\} = [\lceil n/2 \rceil + 2, n - 2].$$

Recall that  $f_2(y_{\lceil n/2 \rceil + 1}) = \lfloor n/4 \rfloor - 1$ , which implies that

$$g(y_{\lceil n/2 \rceil + 1}) = n - \lfloor n/4 \rfloor + f_2(y_{\lceil n/2 \rceil + 1}) = n - 1.$$

This together with  $g(y_{\lceil n/2 \rceil + 2}) = 0$  implies that

$$|g(y_{\lceil n/2 \rceil + 1}) - g(y_{\lceil n/2 \rceil + 2})| = n - 1.$$

It remains to compute the induced edge labels of  $F_1$ . As we can see from the vertex labels computed above, we have  $f_1(u) > f_1(v)$  for all  $uv \in E(F_1)$ , where  $u \in \{x_{2i} : i \in [1, \lceil n/2 \rceil]\}$  and  $v \in \{x_{2i-1} : i \in [1, \lfloor n/2 \rfloor]\}$ . This implies that  $g(u) > g(v)$  and

$$\begin{aligned} |g(u) - g(v)| &= g(u) - g(v) \\ &= n - 1 + (f_1(u) - f_1(v)) \end{aligned}$$

for all  $uv \in E(F_1)$ , where  $u \in \{x_{2i} : i \in [1, \lceil n/2 \rceil]\}$  and  $v \in \{x_{2i-1} : i \in [1, \lfloor n/2 \rfloor]\}$ . By the definition,

$$\{f_1(u) - f_1(v) : uv \in E(F_1)\} = [1, n - 1],$$

since  $f_1$  is an  $\alpha$ -valuation of  $F_1$  with  $f_1(x_1) = 0$ . From this and the last equation on the induced edge labels  $|g(u) - g(v)|$ , where  $uv \in E(F_1)$ , we obtain

$$\{|g(u) - g(v)| : uv \in E(F_1)\} = [n, 2n - 2].$$

It is now immediate that

$$\{|g(u) - g(v)| : uv \in E(F)\} = [1, |E(F)|].$$

Therefore, we conclude that  $\beta_s(F) \leq 2n - 1$  for all integers  $n \geq 12$ , completing the proof.  $\square$

For each integer  $n$  with  $4 \leq n \leq 11$ , we illustrate the construction described in Theorem 7 using Table 1, which contains the vertex labels of  $F_1$ ,  $F_2$  and  $F_3$ . The resulting vertex labeling of  $F$  appears in Table 2.

It is well known that every nontrivial path  $P_n$  is graceful. This together with Corollary 2 implies that  $\beta(mP_n) = mn - 1$  when  $m$  is odd and  $n \geq 2$ . It also follows

TABLE 1. The vertex labeling of  $F_1$ ,  $F_2$  and  $F_3$  for small values of  $n$ 

$n$	$F_1$	$F_2$	$F_3$
4	(0, 3, 1, 2)	(1, 2, 0)	(0)
5	(0, 4, 1, 3, 2)	(2, 1, 3, 0)	(0)
6	(0, 5, 1, 4, 2, 3)	(2, 1, 3, 0)	(0, 1)
7	(0, 6, 1, 5, 2, 4, 3)	(2, 3, 1, 4, 0)	(0, 1)
8	(0, 7, 1, 6, 2, 5, 3, 4)	(3, 0, 4, 2, 1)	(0, 2, 1)
9	(0, 8, 1, 7, 2, 6, 3, 5, 4)	(4, 2, 3, 0, 5, 1)	(0, 2, 1)
10	(0, 9, 1, 8, 2, 7, 3, 6, 4, 5)	(4, 2, 3, 0, 5, 1)	(0, 3, 1, 2)
11	(0, 10, 1, 9, 2, 8, 3, 7, 4, 6, 5)	(2, 5, 3, 4, 0, 6, 1)	(0, 3, 1, 2)

TABLE 2. The vertex labeling of  $F$  for small values of  $n$ 

$n$	$F$
4	(1, 7, 2, 6), (4, 5, 3, 0)
5	(1, 9, 2, 8, 3), (6, 5, 7, 4, 0)
6	(1, 11, 2, 10, 3, 9), (7, 6, 8, 5, 0, 4)
7	(1, 13, 2, 12, 3, 11, 4), (8, 9, 7, 10, 6, 0, 5)
8	(1, 15, 2, 14, 3, 13, 4, 12), (9, 6, 10, 8, 7, 0, 5, 11)
9	(1, 17, 2, 16, 3, 15, 4, 14, 5), (11, 9, 10, 7, 12, 8, 0, 6, 13)
10	(1, 19, 2, 18, 3, 17, 4, 16, 5, 15), (12, 10, 11, 8, 13, 9, 0, 6, 14, 7)
11	(1, 21, 2, 20, 3, 19, 4, 18, 5, 17, 6), (11, 14, 12, 13, 9, 15, 10, 0, 7, 16, 8)

from Theorem 7 and Corollary 3 that  $\beta(mP_n) = mn - 1$  when  $m \equiv 2 \pmod{4}$  and  $n \geq 3$ . Thus, we have the following result.

**Corollary 6.** *For every two positive integers  $m$  and  $n$  such that  $m \not\equiv 0 \pmod{4}$  and  $n \geq 3$ ,*

$$\beta(mP_n) = mn - 1.$$

With the aid of Corollary 6, we have the following result.

**Corollary 7.** *For every positive integer  $m$ ,*

$$\beta(mP_3) = 3m - 1.$$

*Proof.* In light of Lemma 1 and Corollary 6, it suffices to verify that  $\beta(mP_3) \leq 3m - 1$  when  $m \equiv 0 \pmod{4}$ . To do so, let  $F \cong mP_3$  be the forest with

$$V(F) = \{x_i : i \in [1, m]\} \cup \{y_i : i \in [1, m]\} \cup \{z_i : i \in [1, m]\}$$

and

$$E(F) = \{x_i y_i : i \in [1, m]\} \cup \{y_i z_i : i \in [1, m]\},$$

and define the vertex labeling  $f : V(F) \rightarrow [0, 3m - 1]$  such that  $f(x_1) = m$ ;  $f(y_1) = 0$ ;  $f(z_1) = 3m/2$ ; and

$$f(w) = \begin{cases} i - 1 & \text{if } w = x_{2i-1} \text{ and } i \in [2, m/2], \\ m/2 - 1 + i & \text{if } w = x_{2i} \text{ and } i \in [1, m/2], \\ 2m - 1 + i & \text{if } w = y_i \text{ and } i \in [2, m], \\ m - 1 + i & \text{if } w = z_{2i-1} \text{ and } i \in [2, m/2], \\ 3m/2 + i & \text{if } w = z_{2i} \text{ and } i \in [1, m/2]. \end{cases}$$

It remains to observe that

$$\{f(v) : v \in V(F)\} = [0, 3m - 1]$$

and

$$\{|f(u) - f(v)| : uv \in E(F)\} = [m/2, m/2 + |E(F)| - 1].$$

Thus,  $\beta(mP_3) \leq 3m - 1$  when  $m \equiv 0 \pmod{4}$ .  $\square$

We turn our attention to the (strong) beta-number of forests whose components are all stars.

**Theorem 8.** *For every positive integer  $n$ ,*

$$\beta_s(4S_n) = \beta(4S_n) = 4n + 3.$$

*Proof.* In light of Lemma 1, it suffices to show that  $\beta_s(4S_n) \leq 4n + 3$  for every positive integer  $n$ . To do this, define the forest  $F \cong 4S_n$  with

$$V(F) = \{x_i : i \in [1, 4]\} \cup \{y_i^j : i \in [1, 4] \text{ and } j \in [1, n]\}$$

and

$$E(F) = \{x_i y_i^j : i \in [1, 4] \text{ and } j \in [1, n]\},$$

and consider the cases according to the parity of the integer  $n$ .

First, let  $n = 2k - 1$ , where  $k$  is a positive integer, and define the vertex labeling  $f : V(F) \rightarrow [0, 8k - 1]$  such that  $f(x_1) = 0$ ,  $f(x_2) = 8k - 3$ ,  $f(x_3) = 8k - 1$ ,  $f(x_4) = 8k - 2$  and

$$\begin{aligned} f(y_1^j) &= \begin{cases} k & \text{if } j = 1, \\ 2k & \text{if } j = 2, \\ 3k - 3 + j & \text{if } j \in [3, k], \\ 3k + j & \text{if } j \in [k + 1, 2k - 1], \end{cases} \\ f(y_2^j) &= \begin{cases} j & \text{if } j \in [1, k - 1], \\ 4k - 2 & \text{if } j = k, \\ 6k - 3 + j & \text{if } j \in [k + 1, 2k - 1], \end{cases} \\ f(y_3^j) &= \begin{cases} 2k + j & \text{if } j \in [1, k - 1], \\ 4k - 1 & \text{if } j = k, \\ 4k - 1 + j & \text{if } j \in [k + 1, 2k - 1], \end{cases} \\ f(y_4^j) &= \begin{cases} k + j & \text{if } j \in [1, k - 1], \\ 4k & \text{if } j = k, \\ 5k - 2 + j & \text{if } j \in [k + 1, 2k - 1]. \end{cases} \end{aligned}$$

Notice that

$$\begin{aligned} \{f(x_i) : i \in [1, 4]\} &= \{0\} \cup [8k - 3, 8k - 1], \\ \{f(y_1^j) : j \in [1, 2k - 1]\} &= \{k, 2k\} \cup [3k, 4k - 3] \cup [4k + 1, 5k - 1], \\ \{f(y_2^j) : j \in [1, 2k - 1]\} &= [1, k - 1] \cup \{4k - 2\} \cup [7k - 2, 8k - 4], \\ \{f(y_3^j) : j \in [1, 2k - 1]\} &= [2k + 1, 3k - 1] \cup \{4k - 1\} \cup [5k, 6k - 2], \\ \{f(y_4^j) : j \in [1, 2k - 1]\} &= [k + 1, 2k - 1] \cup \{4k\} \cup [6k - 1, 7k - 3]. \end{aligned}$$

This implies that

$$\{f(v) : v \in V(F)\} = [0, 8k - 1],$$

which means that  $f$  is a bijective function. Notice also that

$$\begin{aligned} \left\{ \left| f(x_1) - f(y_1^j) \right| : j \in [1, 2k-1] \right\} &= \{k, 2k\} \cup [3k, 4k-3] \cup [4k+1, 5k-1], \\ \left\{ \left| f(x_2) - f(y_2^j) \right| : j \in [1, 2k-1] \right\} &= [1, k-1] \cup \{4k-1\} \cup [7k-2, 8k-4], \\ \left\{ \left| f(x_3) - f(y_3^j) \right| : j \in [1, 2k-1] \right\} &= [2k+1, 3k-1] \cup \{4k\} \cup [5k, 6k-2], \\ \left\{ \left| f(x_4) - f(y_4^j) \right| : j \in [1, 2k-1] \right\} &= [k+1, 2k-1] \cup \{4k-2\} \cup [6k-1, 7k-3]. \end{aligned}$$

It is now immediate that

$$\{|f(u) - f(v)| : uv \in E(F)\} = [1, |E(F)|].$$

Consequently,  $\beta_s(F) \leq 8k-1$  for every positive integer  $k$ .

Next, let  $n = 2k$ , where  $k$  is a positive integer, and define the vertex labeling  $f : V(F) \rightarrow [0, 8k+3]$  such that  $f(x_1) = 0$ ,  $f(x_2) = 8k+1$ ,  $f(x_3) = 8k+2$ ,  $f(x_4) = 8k+3$  and

$$\begin{aligned} f(y_1^j) &= \begin{cases} k+1 & \text{if } j = 1, \\ 2k+2 & \text{if } j = 2, \\ 3k+j & \text{if } j \in [3, 2k], \end{cases} \\ f(y_2^j) &= \begin{cases} j & \text{if } j \in [1, k], \\ 6k+j & \text{if } j \in [k+1, 2k], \end{cases} \\ f(y_3^j) &= \begin{cases} k+1+j & \text{if } j \in [1, k], \\ 5k+j & \text{if } j \in [k+1, 2k], \end{cases} \\ f(y_4^j) &= \begin{cases} 2k+2+j & \text{if } j \in [1, k], \\ 4k+j & \text{if } j \in [k+1, 2k]. \end{cases} \end{aligned}$$

Notice that

$$\begin{aligned} \{f(x_i) : i \in [1, 4]\} &= \{0\} \cup [8k+1, 8k+3], \\ \{f(y_1^j) : j \in [1, 2k]\} &= \{k+1, 2k+2\} \cup [3k+3, 5k], \\ \{f(y_2^j) : j \in [1, 2k]\} &= [1, k] \cup [7k+1, 8k], \\ \{f(y_3^j) : j \in [1, 2k]\} &= [k+2, 2k+1] \cup [6k+1, 7k], \\ \{f(y_4^j) : j \in [1, 2k]\} &= [2k+3, 3k+2] \cup [5k+1, 6k]. \end{aligned}$$

This implies that

$$\{f(v) : v \in V(F)\} = [0, 8k+3],$$

which means that  $f$  is a bijective function. Notice also that

$$\begin{aligned} \left\{ \left| f(x_1) - f(y_1^j) \right| : j \in [1, 2k] \right\} &= \{k+1, 2k+2\} \cup [3k+3, 5k], \\ \left\{ \left| f(x_2) - f(y_2^j) \right| : j \in [1, 2k] \right\} &= [1, k] \cup [7k+1, 8k], \\ \left\{ \left| f(x_3) - f(y_3^j) \right| : j \in [1, 2k] \right\} &= [k+2, 2k+1] \cup [6k+1, 7k], \\ \left\{ \left| f(x_4) - f(y_4^j) \right| : j \in [1, 2k] \right\} &= [2k+3, 3k+2] \cup [5k+1, 6k]. \end{aligned}$$

It is now immediate that

$$\{|f(u) - f(v)| : uv \in E(F)\} = [1, |E(F)|].$$

Consequently,  $\beta_s(F) \leq 8k + 3$  for every positive integer  $k$ .

Therefore, it follows from the above cases that  $\beta_s(F) \leq 4n + 3$  for every positive integer  $n$ , completing the proof.  $\square$

Now, we obtain the following result from Corollary 3 and Theorem 8.

**Corollary 8.** *For every two positive integers  $m$  and  $n$  such that  $m \equiv 4 \pmod{8}$ ,*

$$\beta(mS_n) = mn + m - 1.$$

Every star  $S_n$  is clearly graceful. This together with Corollary 2 implies that  $\beta(mS_n) = mn + m - 1$  when  $m$  is odd and  $n \geq 1$ . It is also known from Lemma 2 that  $\beta(S_m \cup S_n) = m + n + 1$  when  $mn$  is even. This implies that  $\beta(2S_n) = 2n + 1$  when  $n$  is even. It follows from this and Corollary 3 that  $\beta(mS_n) = mn + m - 1$  when  $m \equiv 2 \pmod{4}$  and  $n$  is even. Combining all these, we have the following result.

**Corollary 9.** *For every two positive integers  $m$  and  $n$  such that  $m$  is odd, or  $m \equiv 2 \pmod{4}$  and  $n$  is even,*

$$\beta(mS_n) = mn + m - 1.$$

## 5. CONCLUSIONS

Ichishima et al. [9] have provided a constructive proof that the strong beta-number of forests is finite. This proof gives a crude upper bound for  $\beta_s(F)$  when  $F$  is a forest. However, we believe that the actual value of  $\beta_s(F)$  is always smaller than the one provided by the proof. The previous explorations on the (strong) beta-numbers of forests in [9] and the results included in this paper lead us to propose the following two conjectures. We first state the weaker of the two conjectures.

**Conjecture 1.** *If  $F$  is a forest of order  $p$ , then  $\beta(F)$  is either  $p - 1$  or  $p$ .*

We actually believe that more is true.

**Conjecture 2.** *If  $F$  is a forest of order  $p$ , then  $\beta_s(F)$  is either  $p - 1$  or  $p$ .*

Of course, if Conjecture 2 is true, so is Conjecture 1 by Lemma 1. Indeed, the truth of Conjecture 2 implies the truth of the following conjecture by Lemma 1.

**Conjecture 3.** *If  $F$  is a forest of order  $p$ , then  $\text{grac}(F)$  is either  $p - 1$  or  $p$ .*

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